CONTENTS

CHAPTER **TOPIC** **PAGE NUMBER**

Chapter 3

Non homogeneous BVP

3.1 Heat Equation

Suppose that r is positive constant. Solve

$$\begin{aligned} ku_{xx} + r = u_t,\ 0 < x < 1,\, t > 0 \\ s.t \\ u(0,t) = 0,\ u(L,t) = u_0,\, t > 0 \\ u(x,o) = f(x),\, 0 < x < 1 \end{aligned}$$

General Solution

$$u(x,t) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)x + \sum_{n=1\infty} A_n\, e^{-kn^2\pi^2 t} \sin\, n\pi x$$

where

$$A_n = 2\int_0^1 \left[f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)\right] \sin\, n\pi x\, dx \tag{3.1}$$

3.2 Exercise 12.6

Problem Solving 3.1. Q1 Use Method1

Solve the equation

$$ku_{tt} = u_t,\, 0 < x < 1,\, t > 0 \tag{3.2}$$

subject to

$$u(0,t) = 100, \ u(1,t) = 100$$
$$u(x,0) = 0$$

Solution to Problem Solving 3.1.
By changing the dependent variable u to a new dependent variable ν by substitution

$$u(x,t) = \nu(x,t) + \psi(x) \tag{3.3}$$

Here $r = 0$, $f(x) = 0$. Substitute (3.3) into (3.2) gives

$$k\nu_{xx} + k\psi_{xx} = \nu_t \tag{3.4}$$

Problem A:

$$k\psi'' = 0 \tag{3.5}$$
$$\psi' = c_1$$
$$\psi = c_1 x + c_2 \tag{3.6}$$

Apply BC:
$u(0,t) = 100 \to \psi(0) = 100 \leftarrow$ Since $\nu(0,t) = 0$,

Eqn (3.6) becomes
$100 = c_1(0) + c_2 \to c_2 = 100$. Thus

$$\psi = c_1 x + 100 \tag{3.7}$$

Now apply BC: $u(1,t) = 100 \to \psi(1) = 100 \leftarrow$ Since $\nu(1,t) = 0$. Thus eqn (3.7) becomes
$100 = \psi(1) = c_1(1) = 100 \to c_1 = 0$. Hence

$$\psi(x) = 100 \tag{3.8}$$

Ic: $\nu(x,0) = f(x) - \psi(x)$. But $f(x) = 0$. Now find A_n by using eqn (3.1).

$$\begin{aligned} A_n &= 2\int_0^1 (f(x) - 100)\, \sin n\pi x \, dx \\ &= 2\left[\frac{100}{n\pi} \cos n\pi x\right]_0^1 \\ &= \frac{200}{n\pi}[(-1)^n - 1] \end{aligned} \tag{3.9}$$

Hence the general solution

$$\begin{aligned} u(x,t) &= \psi(x,t) + \nu(x,t) \\ &= 100 + \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n} e^{-kn^2\pi^2 t} \sin n\pi x \end{aligned}$$

3.3 Wave Equation

$$\begin{aligned} &a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},\ 0 < x < L,\ t > 0 \\ &u(0,t) = 0,\ u(L,t) = 0,\ t > 0 \\ &u(x,0) = f(x),\ \frac{\partial u}{\partial t}\Big|_{t=0} = g(x),\ 0 < x < L \end{aligned}$$

Solution

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left(A_n \sin \frac{n\pi a t}{L} + B_n \sin \frac{n\pi a t}{L} \right) \sin \frac{n\pi x}{L} \\ A_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \\ B_n &= \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx \end{aligned}$$

Problem Solving 3.2. Q9 Exercise 12.6

When a vibrating string is subjected to an external vertical force that varies with the horizontal distance from left to end, the wave equation takes on the form

$$a^2 \frac{\partial^2 u}{\partial x^2} + Ax = \frac{\partial^2 u}{\partial t^2} \tag{3.10}$$

where A is a constant. Solve the partial differential equation subject to

$$u(0,t) = 0,\ u(1,t) = 0,\ t > 0 \tag{3.11}$$

$$u(x,0) = 0,\ \frac{\partial u}{\partial t}\Big|_{t=0} = 0,\ 0 < x < 1 \tag{3.12}$$

Solution to Problem Solving 3.2.

Use $u(x,t) = v(x,t) + \psi(x)$ to change (6.24) to new dependent variable v. Eqn (6.24) becomes

$$a^2 \frac{\partial^2 v}{\partial x^2} + a^2 \psi^{''} + Ax = \frac{\partial^2 v}{\partial t^2}. \tag{3.13}$$

It leads to:
Problem A:

$$a^2 \psi^{''} + Ax = 0,\ \psi(0) = 0,\ \ \psi(1) = 0 \tag{3.14}$$

Problem B:

$$a^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2} \tag{3.15}$$

$$s.t$$

$$v(0,t) = 0,\ \ v(1,t) = 0 \tag{3.16}$$

$$v(x,0) = -\psi(x), \frac{\partial v}{\partial t}\Big|_{t=0} = 0 \tag{3.17}$$

Solve problem A:

$$\psi^{''} = -\frac{A}{a^2} x$$

$$\psi^{'} = -\frac{A}{2a^2} x^2 + c_1$$

$$\psi(x) = \frac{A}{6a^2} x^3 + c_1 x + c_2 \tag{3.18}$$

Apply BC $\psi(0) = 0$, eqn (6.28) gives

$$0 = \psi(0) = -\frac{A}{6a^2}(0) + c_1(0) + c_2 \rightarrow c_2 = 0$$

$$\psi(x) = -\frac{A}{6a^2} x^3 + c_1 x$$

Apply BC $\psi(1) = 0$, the latest eqn becomes

$$0 = \psi(1) = -\frac{A}{6a^2}(1) + c_1(1) \rightarrow c_1 = \frac{A}{6a^2}$$

So,

$$\psi(x) = -\frac{A}{6a^2} x^3 + \frac{A}{6a^2} x \tag{3.19}$$

Solve Problem B by using separation of variables $v(x,t) = X(x)T(t)$. For $\lambda = \alpha^2$, eqn (6.27) that satisfies BC (3.16) and (3.17) gives the solution

$$v(x,t) = \sum_{n=1}^{\infty} (A_n \cos n\pi at + B_n \sin n\pi t) \sin n\pi x \tag{3.20}$$

Where Here $f(x) = 0$,

$$\begin{aligned}
A_n &= 2\int_0^1 (f(x) - \psi(x)) \sin\, n\pi x\, dx \\
&= \frac{2A}{6a^2}\int_0^1 (x^3 - x) \sin\, n\pi x\, dx \\
&= \frac{2A}{6a^2}\left[\int_0^1\, x^3\, \sin\, n\pi x\, dx - \int_0^1\, x\, \sin\, n\pi x\, dx\right] \qquad (3.21)
\end{aligned}$$

. Consider:

$$\begin{aligned}
\int\, x^3\, \sin\, n\pi\, x\, dx &= x^3 \int\, \sin\, n\pi x\, dx - \int [\int\, \sin\, n\pi x, dx] \frac{d}{dx}(x^3) dx \\
&= \frac{x^3}{n\pi}(-\cos\, n\pi x) + \frac{1}{n\pi}\int\, 3x^2\, \cos\, n\pi x\, dx \\
&= -\frac{x^3}{n\pi}\cos\, n\pi x + \frac{3}{n\pi}\left[\frac{x^2}{n\pi}\, \sin\, n\pi x - \int\, \frac{1}{n\pi}(\sin\, n\pi x)\frac{d}{dx}(x^2)dx\right] \\
&= -\frac{x^3}{n\pi}\, \cos\, n\pi x + \frac{3x^2}{n^2\pi^2}\, \sin\, n\pi x - \frac{6}{n^2\pi^2}\left[\int\, x\, \sin\, n\pi x\, dx\right] \\
&= -\frac{x^3}{n\pi}\, \cos\, n\pi x + \frac{3x^2}{n^2\pi^2}\, \sin\, n\pi x \\
&- \frac{6}{n^2\pi^2}\left[x\int\, \sin\, n\pi x\, dx - \int [\int\, \sin\, n\pi x\, dx]\right] \\
&= \frac{x^3}{n\pi}\, \cos\, n\pi x + \frac{3x^2}{n^2\pi^2}\, \sin\, n\pi x \\
&- \frac{6}{n^2\pi^2}\left[-\frac{x}{n\pi}\, \cos\, n\pi x + \frac{1}{n\pi}\int\, \cos\, n\pi x\, dx\right] \\
&= \frac{x^3}{n\pi}\, \cos\, n\pi x + \frac{3x^2}{n^2\pi^2}\, \sin\, n\pi x \\
&+ \frac{6x}{n^3\pi^3}\, \cos\, n\pi x - \frac{6}{n^4\pi^4}\, \sin\, n\pi x \\
\int_0^1\, x^3\, \sin\, n\pi x\, dx &= \left[\frac{x^3}{n\pi}\, \cos\, n\pi x + \frac{3x^2}{n^2\pi^2}\, \sin\, n\pi x + \frac{6x}{n^3\pi^3}\, \cos\, n\pi x - \frac{6}{n^4\pi^4}\, \sin\, n\pi x\right]_0^1 \\
&= -\frac{1}{n\pi}(-1)^n + \frac{6}{n^3\pi^3}(-1)^n \\
&= -\frac{1}{n\pi}(-1)^n + \frac{2}{n^3\pi^3}((-)^n - 1) \\
\int_0^1\, x\, \sin\, n\pi x\, dx &= [x(-\frac{1}{n\pi}\cos\, n\pi x) + \frac{1}{n^2\pi^2}\sin\, n\pi x]_0^1 \\
&= -\frac{1}{n\pi}(-1)^n
\end{aligned}$$

Substitute into (3.42) gives

$$A_n = \frac{2A(-1)^n}{n^3\pi^3 a^2} \tag{3.22}$$

Since $g(x) = 0$, thus $B_n = 0$. From (3.43),

$$v(x,t) = \frac{2A}{a^2\pi^3}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^3}\right]\cos n\pi at \sin n\pi x \tag{3.23}$$

Solution:

$$\begin{aligned} u(x,t) &= \psi(x) + v(x,t) \\ &= \frac{A}{6a^2}(x - x^3) + \frac{2A}{a^2\pi^3}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^3}\right]\cos n\pi at \sin n\pi x \end{aligned}$$

Q2 Dec 2014

Consider the following BVP

$$u_{tt}(x,t) = \frac{1}{25}u_{xx}(x,t) + \sin\frac{x}{2},\ \ 0 < x < 1,\ \ t > 0 \tag{3.24}$$

$$u(0,t) = 0,\ t > 0 \tag{3.25}$$

$$u_x(1,t) = 0,\ t > 0 \tag{3.26}$$

$$u_t(x,0) = 0,\ 0 < x < \pi \tag{3.27}$$

$$u(x,0) = 200\sin\frac{x}{2},\ 0 < x < 1 \tag{3.28}$$

a) Interpret the boundary and initial conditions.
b) Determine $u(x,t)$ for $t > 0$.

Solution ;;

a) IC (3.28)($f(x) = 200\sin\frac{x}{2}$) denotes the initial vertical displacement (transverse vibration) distribution throughout.
IC (3.27) denotes the initial velocity is zero (release from rest).
BC: $u(0,t) = 0$ means that displacement zero at $x = 0$.
BC: $u_x(1,t) = 0$ is called **free-end-condition**
b) Let

$$u(x,t) = v(x,y) + \psi(x) \tag{3.29}$$

Substitute (3.29)into pde gives

$$\begin{aligned} u_{tt} &= \frac{1}{25}u_{xx} + \sin\frac{x}{2} \\ \frac{\partial^2 v}{\partial t^2} &= \frac{1}{25}\left(\frac{\partial^2 v}{\partial x^2} + \psi''\right) + \sin\frac{x}{2} \end{aligned}$$

gives ODE and homogeneous PdE.

$$\frac{1}{25}\psi'' + \sin\frac{x}{2} = 0 \tag{3.30}$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{25}\frac{\partial^2 v}{\partial x^2} \tag{3.31}$$

BC:$u(0,t) = v(0,t) + \psi(0) = 0$ gives $\psi(0) = 0 \leftarrow$ Since $v(0,t) = 0$

BC:=$u_x(1,t) = v_x(1,t) + \psi_x(1) = 0$ gives $\psi_x(1) = 0 \leftarrow$ Since $v_x(1,t) = 0$

Solve eqn (3.30),

$$\begin{aligned}\psi'' &= -25\,\sin\frac{x}{2}\\ \psi' &= -25\,\frac{1}{1/2}(-\cos\frac{x}{2}) + c_1\\ &= 50\,\cos\frac{x}{2} + c_1\\ \psi(x) &= 100\,\sin\frac{x}{2} + c_1 x + c_2\end{aligned}$$

Apply BC:$\psi(0) = 0$ gives
$0 = \psi(0) = 100\,\sin\frac{0}{2} + c_1(0) + c_2 \to c_2 = 0.$ Thus

$$\psi(x) = 100\,\sin\frac{x}{2} + c_1 x \tag{3.32}$$

$$\psi_x(x) = 50\,\cos\frac{x}{2} + c_1 \tag{3.33}$$

Next apply BC: $\psi_x(1) = 0$
$\psi_x(1) = 0 = 50\,\cos\frac{1}{2} + c_1(1) \to c_1 = -50\,\cos\frac{1}{2} = -49.9.$
The solution is

$$\psi(x) = 100\,\sin\frac{x}{2} - 49.9x \tag{3.34}$$

For PDE (3.31) subject to
$v(0,t) = 0,\ v_x(1,t) = 0,\ 0 < x < 1$
$v(x,0) = 200\,\sin\frac{x}{2} - \psi(x)$
Solve (3.31) by using the method of separation of variables. For cases $\lambda = 0$ and $\lambda = -\alpha^2$ give the trivial solution. Now for $\lambda = \alpha^2$, gives
$v(x,t) = \sum_{n=1}^{\infty}(A_n\,\cos n\pi at + B_n\,\sin n\pi at)\sin n\pi x$
Apply IC: $v_t(x,t) = 0, t = 0$

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty}(-n\pi a A_n\,\sin n\pi at + n\pi B_n\,\cos n\pi at)\sin n\pi x$$

$$0 = v_t(x,0) = \sum_{n=1}^{\infty}(-n\pi a B_n)\sin n\pi x$$

Half-range of 0 of sine series will give $B_0 = 0$. Thus

$$v(x,t) = \sum_{i=1}^{\infty} (A_n \cos n\pi at) \sin n\pi x \tag{3.35}$$

Now apply IC: $u(x,0) = 200 \sin \frac{x}{x} \to v(x,0) = 200 \sin \frac{x}{2} - \psi$,

$$v(x,0) = 200 \sin \frac{x}{2} - (100 \sin \frac{x}{2} - 43.9) = \sum_{n=1}^{\infty} A[n] \sin n\pi x$$

Half-range of $100 \sin \frac{x}{2} + 43.9$ of sine series will give A_n,

$$\begin{aligned}
A_n &= \frac{2}{\pi} \int_0^1 (43.9 + 100 \sin \frac{x}{2}) \sin n\pi\, x\, dx \\
&= \frac{2}{\pi} \left[\left(\frac{43.9}{n\pi} \right) (-\cos n\pi x) \right]_0^1 + \frac{200}{\pi} \int_0^1 \frac{1}{2} [\cos(\frac{1}{2} - n\pi)x + \cos(\frac{1}{2} + n\pi)x] dx \\
&= \frac{2}{\pi} \left[\left(\frac{43.9}{n\pi} \right) (-\cos n\pi x) \right]_0^1 + \frac{100}{\pi} \left[\frac{1}{(1/2 - n\pi)} \sin(1/2 - n\pi)x + \frac{1}{(1/2 + n\pi)} \sin(1/2 + n\pi)x \right]_0^1 \\
&= \frac{87.8}{n\pi^2}(1 - (-1)^n) + \frac{1}{(1/2 - n\pi)}(-\sin \frac{1}{2}) + \frac{1}{(1/2 + n\pi)}(-\sin \frac{1}{2})
\end{aligned}$$

$\leftarrow$ use $\cos(A + B) + \cos(A - B) = 2 \sin A \sin B$

$$\begin{aligned}
&= \frac{87.8}{n\pi^2}(1 - (-1)^n) - \frac{1}{\frac{1}{4} - n^2\pi^2} \sin \frac{1}{2} \\
&= \frac{87.8}{n\pi^2}(1 - (-1)^n) - \frac{0.5}{1/4 - n^2\pi^2} \\
&= \frac{87.8}{n\pi^2}(1 - (-1)^n) - \frac{2}{1 - n^2\pi^2}
\end{aligned}$$

Hence the solution is given by

$$u(x,t) = \psi(x) + v(x,t) \tag{3.36}$$

$$= 100 \sin \frac{x}{2} - 43.9 + \sum_{n=1}^{\infty} \left(\frac{87.8}{n\pi^2}(1 - (-1)^n) - \frac{2}{1 - n^2\pi^2} \right) \cos n\pi at \sin n\pi x \tag{3.37}$$

3.4 Laplace Equation

Standard Formula

$$u_{xx} + u_{yy} = 0,\ 0 < x < a,\ 0 < y < b \tag{3.38}$$
$$s.t \tag{3.39}$$

,

$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial u}{\partial x}\Big|_{x=a} = 0, , 0 < y < b \tag{3.40}$$
$$u(x,0) = 0,\ u(x,b) = f(x),\ 0 < x < a \tag{3.41}$$

Solution :

$$u(x,t) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \cos \frac{n\pi}{a} x \tag{3.42}$$
$$A_n = \frac{2}{a \sinh \frac{n\pi}{a} b} \int_0^a f(x) \cos \frac{n\pi}{a} x\, dx \tag{3.43}$$

Q4 Dec 2014 Given

$$u_{xx} + u_{yy} = 0,\ 0 < x < 1,\ 0 < y < 1 \tag{3.44}$$
$$s.t \tag{3.45}$$

,

$$\frac{\partial u}{\partial y}\Big|_{y=0} = 0,\ \frac{\partial u}{\partial y}\Big|_{y=1} = 0, , 0 < x < 1 \tag{3.46}$$
$$u(0,x) = 0,\ u(1,y) = f(y),\ 0 < y < 1 \tag{3.47}$$

Solution : Use the separation of variable method, $u(x,y) = X(x)Y(y)$. Substitute into pde (3.44),

$$X^{''}Y + XY^{''} = 0$$
$$\frac{Y^{''}}{Y} = -\frac{X^{''}}{X} = -\lambda$$

leads to two ODEs.

$$Y^{''} + \lambda Y = 0 \tag{3.48}$$
$$X^{''} - \lambda X = 0 \tag{3.49}$$

For $\lambda = 0$ gives $Y(y) = c_1$ and $\lambda = -\alpha^2$ gives trivial solution. Now for $\lambda = \alpha^2$, and translate BC into $Y^{'} = 0$ and $Y^{'}(1) = 0$, (3.48) becomes

$$Y^{''} + \alpha^2 Y = 0,\ Y^{'}(0) = 0,\ Y^{'}(1) = 0 \tag{3.50}$$

Solve (3.50) gives

$$Y = c_1 \cos \alpha y + c_2 \sin \alpha y \tag{3.51}$$

Apply BC: $Y^{'}(0) = 0$,

$$\begin{aligned} y' &= -c_1\alpha \sin \alpha y + c_2\alpha \cos \alpha y \\ 0 &= c_1\alpha(1) \to c_2 = 0. \end{aligned}$$

So (3.51) becomes

$$Y = c_1 \cos \alpha y \tag{3.52}$$

$$Y^{'} = -c_1\alpha \sin \alpha y \tag{3.53}$$

Use BC: $Y^{'}(1) = 1$. For non trivial $C_1 \neq 0$, $\sin \alpha y = 0 \to \alpha = n\pi$. Thus for $n = 0$, and $n \geq 1$, the eigenfunction of (3.50) are

$Y = c_1$, $n = 0$ and $Y = c_1 \cos n\pi y,\ n = 1, 2, \ldots,$.
Now by interchange $x \leftrightarrow y$, use (3.42),the solution is

$$u(x, y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh n\pi x \cos n\pi y \tag{3.54}$$

Where, from (3.43)

$$A_n = \frac{2}{\sinh n\pi} \int_0^1 f(y) \cos n\pi y \, dy$$

Chapter 4

Orthogonal Series Expansion

- For a certain types of boundary condition s the method of separation of variables and the superposition principle to to an expansion in a trigonometric series that is not a Fourier series.
- To solve the problem in this section , we utilize the concept of orthogonal series expansion or generalized Fourier series.

4.1 Using Orthogonal series expansion

VBP :

The temperature in a rod of a unit length in which heat transfer from its right boundary into a surrounding medium at a constant temperature zero is determine from

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ 0 < x < 1,\ \ t > 0$$

$$u(0,t) = 0,\ \frac{\partial u}{\partial x}\Big|_{x=1} = -hu(1,t),\ h > 0,\ t > 0$$

$$u(x,0) = 1,\ 0 < x < 1$$

Solution

Using separation of variables with $u(x,t) = X(x)T(t)$, we find

$$kX^{''}T = XT^{'}$$

$$\frac{X^{''}}{X} = \frac{1}{k}\frac{T^{'}}{T} = -\lambda$$

leads to the separated equation with BC respectively,

$$X^{''} + \lambda X = 0 \tag{4.1}$$

$$T^{'} + \lambda T = 0 \tag{4.2}$$

with BC: $X(0) = 0$ and $X^{'}(1) + hX(1) = 0$.
Solve (4.1); for $\lambda = 0, -\alpha^2 < 0$ will yield trivial solution.
For $\lambda = \alpha^2$, eqn (4.1) will yield

$$x(x) = c_1 \cos \alpha x + c_2 \sin \alpha x \tag{4.3}$$

Apply BC: $X(0) = 0$,

$$\begin{aligned} X(0) = c_1 \cos(0) + c_2 \sin(0) = 0 \\ c_1 = 0 \\ X(x) = c_2 \sin \alpha x \end{aligned} \tag{4.4}$$

Apply BC: $X^{'}(1) + hX(1) = 0$ on (4.4),

$$\begin{aligned} X^{'}(1) &= \alpha c_2 \cos \alpha(1) \\ hX(1) &= h(c_2 \sin \alpha \\ X^{'}(1) - hX(1) &= c_2(\alpha \cos \alpha + h \sin \alpha = 0 \\ \alpha \cos \alpha + h \sin \alpha = 0 & \\ \tan \alpha &= -\frac{\alpha}{h} \leftarrow \boxed{\text{has an infinite number of roots-see section 11.4}} \end{aligned}$$

If the consecutive positive roots are denoted $a_n, n = 1, 2, \ldots$ then the eigenvalues of the problem are $\lambda_n = \alpha_n^2$ corresponding to eigenfunctions are

$$X(x) = c_2 \sin \alpha_n x \, n = 1, 2, 3 \ldots$$

The solution of DE (4.2) is

$$T(t) = c_3 e^{-k\alpha_n^2 t}$$

and so

$$u_n = X(x)T(t) = A_n \, e^{-k\alpha_n^2 t} \sin \alpha_n x \tag{4.5}$$

$$\textit{and} \tag{4.6}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin \alpha x \tag{4.7}$$

Now apply IC;at t=0, $u(x, 0) = 1, 0 < x < 1$, so that

$$1 = \sum_{n=1}^{\infty} A_n \sin \alpha_n x \tag{4.8}$$

The series (4.8) is not Fourier sine series. It is an expansion of $u(x, 0) = 1$ in term of orthogonal functions. It follows that the set of eigenfunctions $\{\sin \alpha_n x\}$, $n = 1, 2, 3 \ldots$

where $a's$ are defineed by $\tan\,\alpha = -\frac{\alpha}{h}$ is orthogonal with respect to the weight function $p(x) = 1$. So

$$A_n = \frac{\int_0^1 \sin\,\alpha_n x\,dx}{\int_0^1 \sin^2\,\alpha_n^2 x\,dx} \tag{4.9}$$

To evaluate:

$$\begin{aligned}\int_0^1 \sin_n^2 x\,dx &= \frac{1}{2}\int_0^1 (1 - \cos\,2\alpha x)dx \\ &= \frac{1}{2}\left(1 - \frac{1}{2\alpha_n}\sin\,2\alpha_n\right) \qquad (4.10)\end{aligned}$$

Using

$$\sin\,2\alpha_n = 2\,\sin\,\alpha_n\,\cos\,\alpha_n \tag{4.11}$$

$$a_n\,\cos_,\alpha_n = -h\,\sin\,\alpha_n \tag{4.12}$$

(4.10) becomes

$$\int_0^1 \sin^2\alpha_n\,dx = \frac{1}{2}\left(1 - \frac{1}{2\alpha_n}\sin\,2\alpha_n\right) \tag{4.13}$$

$$= \frac{1}{2}(1 - \frac{1}{2\alpha_n}2\,\sin\,\alpha_n\,\cos\,\alpha_n) \tag{4.14}$$

$$= \frac{1}{2}(1 - (h\,\cos^2\,\alpha_n) \tag{4.15}$$

$$= \frac{1}{2h}(h + \cos^2\alpha_n) \tag{4.16}$$

$$\int_0^1 \sin\,\alpha_n x\,dx = \frac{1}{\alpha_n}(1 - \cos\,\alpha_n) \tag{4.17}$$

Consequently (4.9) becomes

$$A_n = \frac{2h(1 - \cos\,\alpha_n)}{\alpha_n(h + \cos^2\,\alpha_n)}$$

Finally the solution of BVP is

$$u(x,t) = 2h\sum_{n=1}^{\infty}\frac{1 - \cos\,\alpha_n}{\alpha_n(h + \cos^2\alpha_n)}e^{-k\alpha_n^2 t}\sin\,\alpha_n x \tag{4.18}$$

4.1.1 Summary

BVP

$$u_t(x,t) = a^2 u_{xx}(x,t),\ 0 < x < L,\ \ t > 0 \tag{4.19}$$
$$u(0,t) = 0,\ t > 0 \tag{4.20}$$
$$u_x(L,t) + hU(L,t) = 0,\ t > 0 \tag{4.21}$$
$$u(x,o) = f(x),\ 0 < x < L \tag{4.22}$$

General solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{-\left(\frac{z_n \alpha}{L} t\right)^2} \sin \frac{z_n x}{L} \leftarrow \boxed{z_n = \alpha_n}$$

Q1 Exercises 12.7 In example 1 find the temperature $u(x,t)$ when the left end of the rod is insulated.

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ \ 0 < x < 1,\ t > 0 \tag{4.23}$$
$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial u}{\partial x}\Big|_{x=1} = -hu(1,t),\ h > 0,\ t > 0 \tag{4.24}$$
$$u(x,0) = 1,\ \ 0 < x < 1 \tag{4.25}$$

Solution :

Let $u(x,t) = X(x)T(t)$. The substitute into (4.23) gives

$$X^{''} + \lambda X = 0 \tag{4.26}$$
$$T^{''} + k\lambda T = 0 \tag{4.27}$$
$$X^{'}(0) = 0 \text{ and } X(0) = 0,\ X^{'}(1) + h(X(1) = 0 \tag{4.28}$$

Solve (4.26):
For $\lambda = 0$ and $\lambda = -\alpha^2 < 0$ give $u(x,t) = 0$ (trivial solution). For $\lambda = \alpha^2$,

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x \tag{4.29}$$

Apply BC $X^{'}(0) = 0$ gives

$$X^{'}(0) = -c_1\alpha \sin \alpha(1) + c_2\alpha \cos \alpha(1) = 0 \rightarrow C_2 = 0$$

So

$$X(x) = c_1 \cos \alpha\, x \tag{4.30}$$

Apply the second BC of (4.28) to (4.30) yields

$$\begin{aligned} X^{'}(x) &= -c_1\alpha \sin \alpha x \\ X^{'}(1) &= -c_1\alpha \sin \alpha \\ hX(1) &= hc_1 \cos \alpha \\ X^{'}(1) + hX(1) &= -c_1\alpha \sin \alpha + c_1 h \cos \alpha = 0 \\ &= c_1(-\alpha \sin \alpha + h \cos \alpha) = 0 \\ -\alpha \sin \alpha + h \cos \alpha = 0 & \\ \text{or} & \\ \tan \alpha &= \frac{h}{\alpha} \end{aligned}$$

The last eqn has an infinite number of roots. If the positive roots are denoted by α_n, $n = 1, 2, 3 \ldots$, the the eigenvalues of the problem are $\lambda_n = \alpha_n^2$. The corresponding eigenfunctions are

$$X(x) = c_1 \cos \alpha_n x, \;\; n = 1, 2, 3 \ldots \tag{4.31}$$

Solve ODE (4.27) gives $T = e^{-k\alpha_n^2 t}$, and so

$$u_n = XT = A_n e^{-k\alpha_n^2 t} \cos \alpha_n x$$

and

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \cos \alpha_n x \tag{4.32}$$

Now apply IC $u(x, 0) = 1$. At $t = 0$ so that

$$u(x,0) = 1 = \sum_{n=1}^{\infty} A_n \cos \alpha_n x \tag{4.33}$$

The series in (4.33) is an expansion of $u(x, 0) = 1$ in terms of orthogonal function. It follows that the set eigenfunctions $\{\cos \alpha_n x\}$, $n = 1, 2, \ldots$ where $\alpha_n' s$ are defined by

$\tan\,\alpha = h/\alpha$ is orthogonal with respect to $p(x) = 1$. It follows that

$$\begin{aligned}
A_n &= \frac{\int_0^1 \cos\,\alpha_n x\,dx}{\int_0^1 \cos^2\,\alpha_n x\,dx} \\
\int_0^1 \cos\,\alpha_n x\,dx &= \frac{1}{\alpha_n}[\sin\,\alpha_n x]_0^1 \\
&= \frac{1}{\alpha_n}\sin\,\alpha_n \\
\int_0^1 \cos^2\alpha x\,dx &= \frac{1}{2}\int_0^2 (1+\cos\,2\alpha_n x)dx \\
&= \frac{1}{2}\left[x + \frac{1}{2\alpha_n}\sin\,2\alpha_n x\right]_0^1 \leftarrow \boxed{\cos\,x^2 = \tfrac{1+\cos\,2x}{2}} \\
&= \frac{1}{2}\left[1 + \frac{1}{2\alpha_n}\sin\,2\alpha_n\right] \\
&= \frac{1}{2}\left[1 + \frac{1}{2\alpha_n}2\,\sin\,\alpha\,\cos\,\alpha\right] \leftarrow \boxed{\text{use } \cos\,2x = 2\,\sin\,x\,\cos\,x} \\
&= \frac{1}{2}\left[1 + \frac{1}{\alpha}\sin\,\alpha_n(\frac{\alpha}{h}\sin\,\alpha_n)\right] \leftarrow \boxed{\cos\,\alpha = \tfrac{\alpha}{h}\sin\,\alpha} \\
&= \frac{1}{2h}(h+\sin^2\alpha) \\
&= \frac{2h}{\alpha_n}\frac{\sin\,\alpha_n}{h+\sin^2\,\alpha_n}
\end{aligned}$$

Hence the general solution is

$$u(x,t) = 2h\sum_{n=1}^{\infty}\frac{\sin\,\alpha_n}{\alpha_n(h+\sin^2\,\alpha_n)}e^{-k\alpha_n^2 t}\cos\,\alpha_n x$$

4.2 Past Sem Paper

Problem Solving 4.1. Q3 JUN 2012

Consider the following boundary-value problem:

$$\begin{aligned}
u_t(x,t) &= u_x x(x,t) + 2,\ 0 < x < 1,\ t > 0 \\
u(0,t) &= 0,\ u_x(1,t) + u(1,t) = 0,\ t > 0 \\
u(x,t) &= x + 1,\ 0 < x < 1
\end{aligned}$$

Chapter 5

Higher Dimensional Problems

5.1 Two-Dimensional Linear PDE that represent Temperature

Given

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=\frac{\partial u}{\partial t},\ 0<x<b,\ 0<y<c,\ t>0$$

$s.t$

$$\begin{aligned}
u(0,y,t)=0,\ u(b,y,t)=0,\ 0<y<c,\ \ t>0\\
u(x,0,t)=0,\ u(x,c,t)=0,\ 0<y<c,\ t>0\\
u(x,y,0)=f(x,y),\ 0<x<b,\ \ 0<y<c
\end{aligned}$$

Solution

Apply separation of variables

$$u(x,y,t)=X(x)Y(y)T(t) \tag{5.1}$$

Substitute (**??**) into PDE gives

$$k(X^{''}YT+XY^{''}T)=XYT^{'}$$

or

$$\frac{X^{''}}{X}=-\frac{Y^{''}}{Y}+\frac{T^{'}}{kT} \tag{5.2}$$

Both sides of eqn (5.2) must equal to a constant $-\lambda$:

$$X^{''}+\lambda X=0 \tag{5.3}$$

$$\frac{Y^{''}}{Y}=\frac{T^{'}}{KT}+\lambda \tag{5.4}$$

Introduce another separation constant $-\mu$ in (5.4) becomes

$$\frac{Y''}{Y} = -\mu \text{ and } \frac{T'}{kT} + \lambda = -\mu$$
$$Y'' + \mu Y = 0 \text{ and } T' + k(\lambda + \mu)T = 0 \tag{5.5}$$

Now homogenoues BC:

$$\left.\begin{array}{ll} u(0,y,t) = 0, & u(b,y,t) = 0 \\ u(x,0,t) = 0, & u(x,c,t) = 0 \end{array}\right\} \rightarrow \left\{\begin{array}{ll} X(0) = 0, & X(b) = 0 \\ Y(0) = 0, & Y(c) = 0 \end{array}\right.$$

Thus we have two Sturm-Liouville problems:

$$X'' + \lambda X = 0,\ X(0) = 0,\ \ X(b) = 0 \tag{5.6}$$
$$Y'' + \mu Y = 0,\ Y(0) = 0,\ Y(c) = 0 \tag{5.7}$$

Solve (5.6):-
Case 1: $\lambda = 0$, $\mu = 0$:

$$X(x) = c_1 x + c_2 \tag{5.8}$$

Apply BC, $X(0) = 0$ on (5.8) gives

$$X(0) = 0 = c_1(0) + c_1 \rightarrow c_2 = 0$$

So $X(x) = c_1 x$. Apply BC, $X(b) = 0$ gives $0 = c_1(0) \rightarrow c_1 = 0$ Similarly for (5.7) will give $Y(y) = 0$ when $\mu = 0$.
Case 2: $\lambda = -\alpha^2 < 0$, $\mu = -\alpha^2 < 0$:
Eqns (5.6) and (5.7) become
$X(x) = c_3 \cosh \alpha x + c_4 \sinh \alpha x$ and $Y(y) = c_5 \cosh \alpha y + c_6 \sinh \alpha y$ will give the trivial solution when apply BCs.
<u>Case 3:</u>$\lambda = \alpha^2 > 0$, $\mu = \alpha^2 > 0$:
The solution of eqns (5.6) and (5.7) are

$$X(x) = c_7 \cos \alpha x + c_8 \sin \alpha x \tag{5.9}$$
$$Y(y) = c_9 \cos \alpha x + c_{10} \sin \alpha x \tag{5.10}$$

When apply BC on (5.9) gives

$$X(0) = 0 = c_7(1) \rightarrow c_7 = 0$$
$$X(x) = c_8 \sin \alpha x$$

Apply BC, $Y(b) = 0$, give the nontrivial solution: $c_8 \neq 0$, $\sin \alpha(b) = 0 \rightarrow \alpha = \frac{n\pi}{b}$. So the eigenvlue, $\lambda_m = \frac{m^2\pi^2}{b^2}$ and the corresponding eigenfunction

$$X(x) = c_8 \sin \frac{m\pi}{b} x \tag{5.11}$$

Similarly , the eigenvlue and eigenfunction of (5.10) are

$$\lambda_n = \frac{n^2\pi^2}{c^2} \tag{5.12}$$
$$Y(y) = c_{10}\,\sin\,\frac{n\pi}{c} \tag{5.13}$$

Now solve (5.5),

$$T^{'} + k(\frac{m^2\pi}{b^2} + \frac{n^2\pi^2}{c^2})T = 0$$
$$T = c_{11}e^{-k(\frac{m^2\pi}{b^2}+\frac{n^2\pi^2}{c^2})t}$$

A product solution of the two-dimensional heat equation,

$$u_{mn}(x, y, t) = A_{mn}e^{-(\frac{m^2\pi}{b^2}+\frac{n^2\pi^2}{c^2})t}\,\sin\,\frac{m\pi}{b}x\,\sin\,\frac{n\pi}{c}y$$

where A_{mn} is an arbitrary constant.
By the superposition principle, the solution,

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}e^{-k(\frac{m^2\pi}{b^2}+\frac{n^2\pi^2}{c^2})t}\,\sin\,\frac{m\pi}{b}x\,\sin\,\frac{n\pi}{c}y \tag{5.14}$$

Now apply IC; at $t = 0$, eqn (5.15) gives

$$u(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\,\sin\,\frac{m\pi}{b}x\,\sin\,\frac{n\pi}{c}y$$
$$A_{mn} = \frac{4}{bc}\int_0^c\int_0^b f(x, y)\,\sin\,\frac{m\pi}{b}x\,\sin\,\frac{n\pi}{c}y\,dx\,dy \tag{5.15}$$

Thus the solution of the BVP consists of (5.14) with A_{mn} given by (5.15)

5.2 BVP subjected to the following BC

$$u_t(x, y, t) = a^2[u_{xx}(x, y, t) + u_{yy}(x, y, t)],\ 0 < x < b,\ 0 < y < c,\ t > 0$$

$s.t$

$$u_x(0, y, t) = u_x(b, y, t) = 0,], 0 < y < c,\ t > 0$$
$$u_y(x, 0, t) = u)y(x, c, t) = 0,\ 0 < x < b,\ t > 0$$
$$u(x, y, 0) = f(x, y),\ 0 < x < b,\ 0 < y < c$$

Solution

$$u(x,y,t) = A_{00} + \sum_{m=1}^{\infty} A_{m0} \cos\left(\frac{m\pi x}{b}\right) + \sum_{n=1}^{\infty} A_{0n} \cos\left(\frac{n\pi y}{c}\right)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \cos\left(\frac{m\pi x}{b}\right) \cos\left(\frac{n\pi y}{c}\right)$$
$$A_{00} = \frac{1}{bc}\int_0^c$$

5.3 Suggested answers

5.3.1 Q4 Jun 2014

Problem Solving 5.1. Consider the following two-dimensional heat equation

$$u_t(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t),\ 0 < x < 2\pi,\ 0 < y < 2\pi,\ t > 0$$
$$u(0,y,t) = u(2\pi,y,t) = 0,\ 0 < y < 2\pi$$
$$u(x,0,t) = u(x,2\pi,t) = 0,\ 0 < x < 2\pi$$
$$u(x,y,0) = x\, sin\, y,\ 0 < y < 2\pi$$

a) Interpret the boundary-value problem above.

b) Determine the temperature distribution $u(x,y,t)$.

Solution

a) BVP describe the problem of temperature over the rectangle $0 \le x \le \pi$ by $0 \le y \le \pi$ whose the initial temperature is x sin y throughout and boundaries are held at temperature zero for time $t > 0$.

b) By using the method of the separation of variable $u(x,y,t) = X(x)Y(y)T(t)$ and consider the $\lambda = \alpha^2 > 0$ and $\mu = \alpha^2 > 0$ we obtain the aigenvalues and theirs corresponding eigenfunctions

$$\lambda_m = \frac{m\pi}{2\pi} = \frac{m}{2}\ \ \mu_n = \frac{n\pi}{2\pi} = \frac{n}{2}$$

and the corresponding eigenfunctions

$$X(x) = c_1 \sin\frac{m}{2}x,\ Y(y) = c_2 \sin\frac{n}{2}y$$

respectively, where $m = 1, 2, 3 \ldots$ and $n = 1, 2, 3, \ldots$
A product solution of the 2-D heat equation that satisfies BC

$$u_{mn} = A_{mn} e^{-(\frac{m^2}{4} + \frac{n^2}{4})} \sin \frac{m}{2} x \sin \frac{n}{2} y \tag{5.16}$$

where A_{mn} is an arbitrary constant.
By the superposition principle in the form of a double sum

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-(\frac{m^2+n^2}{4})} \sin \frac{m}{2} x \sin \frac{n}{2} y \tag{5.17}$$

To find A_{mn}:
At $t = 0$

$$x \sin y = u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m}{2} x \sin \frac{n}{2} y$$

We can find the coefficient A_{mn} by using the double integration,

$$\begin{aligned}
A_{mn} &= \frac{4}{2\pi 2\pi} \int_0^{2\pi} \int_0^{2\pi} x\, sin\, y \sin \frac{m}{2} x \sin \frac{n}{2} y \, dx \, dy \\
&= \frac{1}{\pi^2} \int_0^{2\pi} \left[-\frac{2x}{m} \cos \frac{m}{2} x + \frac{4}{m^2} \sin \frac{m}{2} x \right]_0^{2\pi} \sin \frac{m}{2} y \, dy \\
&= -\frac{4}{\pi} (-1)^m \int_0^{2\pi} \sin \frac{m}{2} y \, dy \\
&= -\frac{4}{m\pi} (-1)^m \left[-\frac{2}{m} \cos \frac{m}{2} y \right]_0^{2\pi} \\
&= \frac{8}{m^2 \pi} (-1)^m ((-1)^m - 1)
\end{aligned}$$

The solution is given by (5.17) where $A_{mn} = \frac{8}{m^2\pi}(-1)^m((-1)^m - 1)$,

Chapter 6

VBP In Polar coordinate

6.1 Exercise 13.1/ Q5.Jun 2012

Problem Solving 6.1. Q13.9

Solve VBP

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ 0 < \theta < 2\pi,\ a < r < b$$
$$u(a,\theta) = f(\theta),\ u(b,\theta) = 0$$

Solution

a) Defining $u(r,\theta) = R(r)\Theta(\theta)$ and separating variable gives,

$$R^{''}\Theta + \frac{1}{r}R^{'}\Theta + \frac{1}{r^2}R\Theta^{''} = 0$$
$$r^2R^{''}\Theta + rR^{'}\Theta + R\Theta^{''} = 0 \leftarrow \boxed{\text{multiply by } r}$$
$$r^2R^{''}\Theta + rR^{'}\Theta = -R\Theta^{''}$$
$$\frac{r^2R^{''} + rR^{'}}{R} = -\frac{\Theta^{''}}{\Theta} = \lambda$$

The separate equations are

$$r^2R^{''} + rR^{'} - \lambda R = 0 \tag{6.1}$$
$$\Theta^{''} + \lambda\Theta = 0 \tag{6.2}$$

Solve (6.2) of the problem

$$\Theta^{''} + \lambda\Theta = 0,\ \ \Theta(\theta) = \Theta(\theta + 2\pi) \tag{6.3}$$

For $\lambda = 0$:

$$\Theta(\theta) = c_1 + c_2\theta \tag{6.4}$$

For $\lambda = -\alpha < 0$.

$$\Theta(\theta) = c_1 \cosh\ \alpha\theta + c_2 \sinh\ \alpha\theta \tag{6.5}$$

For $\lambda = \alpha^2 > 0$,

$$\Theta(\theta) = c_1 \cos\alpha\theta + c_2 \sin\alpha\theta \tag{6.6}$$

Solution (6.4) is nonperiodic unless $C_2 = 0$. Similarly, solution (6.5) is non periodic unless $c_1 = 0$ and $c_2 = 0$. Solution (6.6) will be 2π-periodic if we take $\alpha = n$. the eigenvalues of (6.1) are then $\lambda_0 = 0$ and $\lambda_n = n^2$, $n = 1, 2, 3.\ldots.$ If we correspond $\lambda_0 = 0$
with $n = 0$, the eigenfunctions of (6.1) are

$$\Theta(\theta) = c_1,\ n = 0\text{and } \Theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta,\ n = 1, 2, 3, \ldots \tag{6.7}$$

Now solve (6.1).
For $\lambda_0 = 0$, $n = 0$ Let $R = r^m$, $R' = mr^{m-1}$ and $R'' = m(m-1)r^{m-2}$, substituting into (6.1),

$$\begin{aligned} r^2m(m-1)r^{m-2} + rmr^{m-1} &= 0 \\ (m(m-1) + m)r^{m-1} &= 0 \\ m &= 0, 0 \end{aligned}$$

So, the solution is

$$R(r) = c_1 + c_2 \ln r \tag{6.8}$$

For $\lambda = n^2$,

$$\begin{aligned} r^2m(m-1)r^{m-2} + rmr^{m-1} - n^2r^m &= 0 \\ m^2 - n^2 &= 0 \\ m &= n, -n \end{aligned}$$

The solution is

$$R(r) = c_1r^n + c_2r^{-n} \tag{6.9}$$

Apply BC, $u(b, \theta) = R(b)\Theta(\theta) = 0$. Since $\Theta(\theta) \neq 0$, thus $R(b) = 0$ in (6.8) and (6.9) gives

$$R(b) = c_1 + c_2 \ln b = 0 \leftrightarrow c_1 = -c_2 \ln b \tag{6.10}$$

$$So\ R(r) = c_2(\ln r - \ln b) = c_2 \ln \frac{r}{b} \leftarrow \boxed{\text{from (6.8)}} \tag{6.11}$$

So from (6.7) and (6.11), a product solution when $\lambda_0 = 0$ is

$$u_0(r,\theta) = R(r)\Theta(\theta) \tag{6.12}$$
$$= \left(c_2 \ln \frac{r}{b}\right) c_1 \tag{6.13}$$
$$= A_0 \ln \frac{r}{b} \leftarrow \boxed{c_1 c_2 = A_0} \tag{6.14}$$

Now apply BC; $R(b) = 0$ on (6.9) give

$$R(b) = c_1 b^n + c_2 b^{-n} = 0 \leftrightarrow c_1 = -c_0 b^{-n},\ c_2 = c_0 b^n \tag{6.15}$$
$$So\ R(r) = -c_0 b^{-n} r^n + c_0 b^n r^{-n} \tag{6.16}$$
$$= c_0 \left(\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right) \tag{6.17}$$

A product solution when $\lambda_n = n^2,\ n = 1, 2, \ldots$ then

$$u_n(r,\theta) = c_0 \left(\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right) (c_1 \cos n\theta + c_2 \sin n\theta) \tag{6.18}$$

Form (6.14) and (6.18) and by superposition principle then solution is,

$$u(r,\theta) = u_n(r,\theta) + u_n(r,\theta) \tag{6.19}$$
$$= A_0 \ln \frac{r}{b} + \sum_{i=1}^{n} c_0 \left(\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right) (c_1 \cos n\theta + c_2 \sin n\theta)$$
$$= A_0 \ln \frac{r}{b} + \sum_{i=1}^{n} \left(\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right) (A_n \cos n\theta + B_n \sin n\theta) \leftarrow \boxed{c_0 c_1 = A_n \text{ and } c_0 c_2 = B_n} \tag{6.20}$$

b) If $a = 1$, $b = 2$ and $f(\theta) = \cos\theta$, determine the specific solution $u(r,\theta)$:
Apply BC $u(a,\theta) = f(\theta)$ gives

$$f(\theta) = u(a,\theta) = A_0 \ln \frac{a}{b} + \sum_{i=1}^{n} \left(\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right) (A_n \cos n\theta + B_n \sin n\theta) \tag{6.21}$$
$$\tag{6.22}$$

where

$$
\begin{aligned}
A_0 \ln \frac{1}{2} &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta \\
A_0 &= \frac{1}{2\pi \ln \frac{1}{2}} \int_0^{2\pi} \cos\, \theta\, d\theta \\
&= \frac{1}{2\pi} [\sin\, \theta]_0^{2\pi} \\
&= 0 \\
A_n \left(\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n \right) &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, \cos\, n\theta\, d\theta \\
A_n &= \frac{1}{\left(\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right)\pi} \int_0^{2\pi} f(\theta)\, \cos\, n\theta\, d\theta \\
&= \frac{1}{\left(\left(\frac{2}{1}\right)^n - \left(\frac{1}{2}\right)^n\right)\pi} \int_0^{2\pi} \cos\, \theta\, \cos\, n\theta\, d\theta \\
&= \frac{2^1}{(4^1 - 1)\pi} \int_0^{2\pi} \cos^2\, \theta\theta \leftarrow \boxed{n = 1. \text{ If } n \neq 1, \text{ then orthogonal function, } 0} \\
&= \frac{2}{3\pi} \int_0^{2\pi} \frac{1}{2}(1 + \cos\, 2\theta)\, d\theta \\
&= \frac{1}{3\pi} [\theta + \frac{1}{2} \sin\, 2\theta]_0^{2\pi} \\
&= \frac{1}{3\pi}(2\pi) = \frac{2}{3} \\
B_n \left(\left(\frac{2}{1}\right)^n - \left(\frac{1}{2}\right)^n \right) &= \frac{1}{2\pi} \int_0^{2\pi} \cos\, \theta\, \sin\, n\theta\, d\theta \\
B_n &= \frac{1}{\left(\left(\frac{2}{1}\right)^n - \left(\frac{1}{2}\right)^n\right)\pi} \int_0^{2\pi} \cos\, \theta\, \sin\, n\theta\, d\theta \\
&= \frac{2^n}{(4^n - 1)\pi} \int_0^{2\pi} \frac{1}{2}(\sin(1+n)\theta - \sin(1-n)\theta)\, d\theta \\
&= \frac{2^{n-1}}{(4^n - 1)\pi} \left[\frac{1}{1+n}(-\cos(1+n)\theta + \frac{1}{1-n} \cos(1-n)\theta \right]_0^{2\pi} \\
&= \frac{2^{n-1}}{(4^n - 1)\pi} \left[\frac{-1}{1+n}(\cos(1+n)2\pi - 1) + \frac{1}{1-n}(\cos(1-n)2\pi - 1) \right]
\end{aligned}
$$

Consider

$$\begin{aligned}\cos(1+n)2\pi &= \cos(2\pi)\cos(2n\pi) - \sin(2\pi)\sin(2n\pi)\\ &= 1\\ \cos(1-n)2\pi &= \cos(2\pi)\cos(2n\pi) + \sin(2\pi)\sin(2n\pi)\\ &= 1\end{aligned}$$

So

$$B_n = 0$$

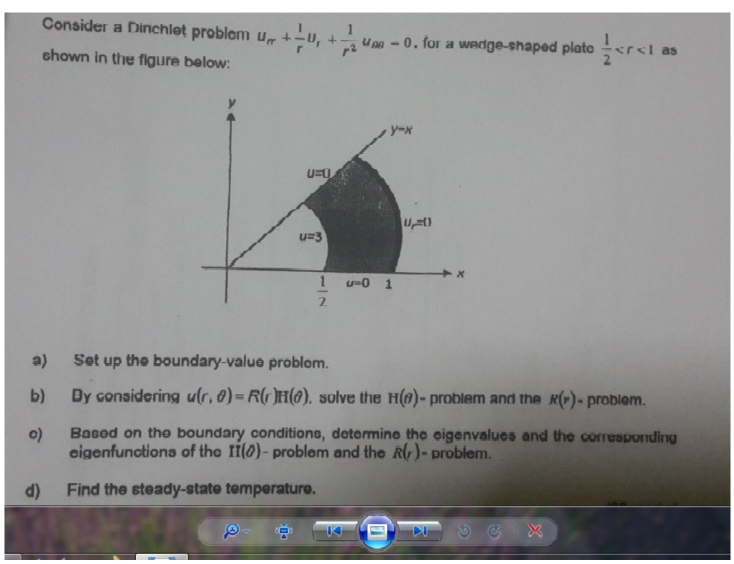
The specific solution

$$u(r,\theta) = \frac{2}{3}\sum_{n=1}^{n}\left(2^n - 2^{-n}\right)\cos n\theta \tag{6.23}$$

6.2 Wedge-shape plate

- The Dirichlet Problem. The problem is to find a harmonic function u inside a domain D so that the values of u are prescribed on the boundary ∂D of D ($u = f$ is given on the boundary D).
- The Neumann Problem. The problem is to find a harmonic function$u$ inside the domain D so that the normal derivatives of u, (i.e. u) are prescribed on the boundary ($\frac{\partial u}{\partial \eta} = g$ on ∂D.) Recall that the normal derivative at a point.

6.2.1 Q5 Jan 2012


Consider a Dirichlet problem $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$, for a wedge-shaped plate $\frac{1}{2} < r < 1$ as shown in the figure below:

a) Set up the boundary-value problem.

b) By considering $u(r,\theta) = R(r)\Theta(\theta)$, solve the $\Theta(\theta)$- problem and the $R(r)$- problem.

c) Based on the boundary conditions, determine the eigenvalues and the corresponding eigenfunctions of the $\Theta(\theta)$- problem and the $R(r)$- problem.

d) Find the steady-state temperature.

Solution

a) BVP

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,\ \frac{1}{2} < r < 1 \tag{6.24}$$
$$s.t \tag{6.25}$$
$$u(r,0) = 0,\ \frac{1}{2} < r < 1 \tag{6.26}$$
$$u(\frac{1}{2},\theta) = 3,\ 0 < \theta < \frac{\pi}{4} \tag{6.27}$$
$$u_r(1,\theta) = 0,\ 0 < \theta < \frac{\pi}{4} \tag{6.28}$$
$$u(r,\frac{\pi}{4}) = 0,\ \frac{1}{2} < r < 1 \tag{6.29}$$

b) By considering $u(r,\theta) = R(r)H(\theta)$, solve the $H(\theta)$-problem and the $R(r)$-problem. From (6.24),

$$\begin{aligned} R^{''}H + \frac{1}{r}R^{'}H + \frac{1}{r^2}RH^{''} &= 0 \\ r^2R^{''}H + rR^{'}H + RH^{''} &= 0 \\ r^2R^{''}H + rR^{'}H &= -RH^{''} \\ \frac{r^2R^{''} + rR^{'}}{R} &= -\frac{H^{''}}{H} = \lambda \end{aligned}$$

leads to two ODEs

$$r^2R^{''} + rR^{'} - \lambda R = 0 \tag{6.30}$$
$$H^{''} + \lambda H = 0 \tag{6.31}$$

We are looking for

$$H^{''} + \lambda H = 0, \tag{6.32}$$

The three possibles general solution of (6.32),

$$H(\theta) = c_1 + c_2\theta,\ \lambda = 0 \tag{6.33}$$
$$H(\theta) = c_1 \cosh \alpha\theta + c_2 \sinh \alpha\theta,\ \lambda = -\alpha^2 < 0 \tag{6.34}$$
$$H(\theta) = c_1 \cos \alpha\theta + c_2 \sin \alpha\theta,\ \lambda = \alpha^2 > 0 \tag{6.35}$$

The solution of Cauchy-Euler (6.30) are as follows. Let $R = r^m, R^{'} = mr^{m-1}, R^{''} =$

$m(m-1)$. Substitute into (6.30) and for $\lambda = 0$, becomes

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} = 0$$
$$m(m-1)r^m + mr^m =$$
$$m^2 r^m = 0$$
$$m = 0, 0 \leftarrow \boxed{\text{equal real, } R(r) = c_1 r^m + c_2 r^m \ln r}$$

Thus the solution is

$$R(r) = c_1 + c_2 \ln r \tag{6.36}$$

For $\lambda_n = \alpha^2$, (6.30) becomes

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - \alpha^2 r^m = 0$$
$$(m^2 - \alpha^2)r^m = 0$$
$$m = \alpha, -\alpha \leftarrow \boxed{\text{real and different olution} R(r) = c_1 r^{m_1} + c_2 r^{-m_2}}$$

Thus the solution is

$$R(r) = c_1 r^{\alpha} + c_2 r^{-\alpha} \tag{6.37}$$

c) Apply BC on H-problem:

The BC (6.26) and (6.29)w5 together with (6.30) constitute a regular Sturm-Liouville problem

Now apply BC (6.26) and (6.29): $u(r, 0) = R(r)H(0) = 0$. Since $R(r) \neq 0$, so $H(0) = 0$. $u(r, \pi/4) = 0 \to H(\frac{\pi}{4}) = 0$.
The regular Sturm-Liouville problem:

$$H'' + \lambda H = 0, \; H(0) = 0, \; H(\frac{\pi}{4}) = 0. \tag{6.38}$$

From (6.33) gives

$$H(0) = c_1 + c_2(0) = 0 \to c_2 = 0$$
$$So\ H(\theta) = c_1 \tag{6.39}$$
$$H(\frac{\pi}{4}) = c_1 = 0 \to c_1 = 0 \tag{6.40}$$

Thus $u(r, \theta) = 0$ when $\lambda = 0$.
From (6.34),

$$u(r, 0) = c_1 \cosh(0) + c_1 \sinh(0) = 0$$
$$= c_1 = 0 \to c_1 = 0.$$
$$So\, u(r, \theta) = c_2 \sinh \alpha\theta \tag{6.41}$$

From BC (6.29): $u(r, \frac{\pi}{4}) = R(r)H(\frac{\pi}{4}) = 0 \rightarrow H(\frac{\pi}{4}) = 0$.

$$u(r, \frac{\pi}{4}) = c_1 \sinh \alpha\frac{\pi}{4}$$

This solution is unbounded and nonperiodic unless $c_1 = 0$, So the solution is trivial. Now apply BC (6.26) and (6.29) on (6.35),

$$\begin{aligned} u(r, 0) &= c_1 \cos(0) + c_2 \sin(0) = 0 \\ &= c_1 = 0 \rightarrow c_1 = 0 \end{aligned}$$

For nontrivial solution $c_2 \neq 0$, so $\sin \alpha\frac{\pi}{4} = 0 \rightarrow \alpha = 4n$. So the problem (6.38) possesses eigenvalues $\lambda_n = 4n, \ n = 1, 2, \ldots$. The eigenfunction is

$$H(\theta) = c_2 \sin 4n\theta, \ \ n = 1, 2, 3, \ldots \tag{6.42}$$

Apply BC on R-problem:
Transform BC (6.27) give $R^{'}(1) = 0$. From (6.36)

$$\begin{aligned} R(r) &= c_1 + c_2 \ln r \\ R^{'}(r) &= \frac{c_2}{r} \\ R^{'}(0) &= \frac{c_2}{1} \rightarrow c_2 = 0 \end{aligned}$$

$R(r) = C_1$, So the solution is trivial.
Since we want $R(r)$ to be bounded as $r \rightarrow 0$, so eqn (6.37) we find that $c_2 = 0$ and $\lambda = \alpha = 4n$. Thus (6.37) becomes

$$R(r) = c_1 r^{4n} \tag{6.43}$$

d) From (6.42) and (6.43) we obtain a product solution

$$\begin{aligned} u_n(r, \theta) &= R(r)H(\theta) \\ &= c_1 r^{4n}(c_2 \sin 4n\theta) \\ &= A_n r^{4n} \sin 4n\theta \end{aligned}$$

And by using the superposition principle we obtain the solution of the steady-state temperature,

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^{4n} \sin 4n\theta \tag{6.44}$$

Now find A_n: apply BC $u(1/2,\theta)=3$

$$u(\frac{1}{2},\theta)=3=\sum_{n=1}^{\infty}A_n\left(\frac{1}{2}\right)^{4n}$$

$$\begin{aligned}A_n\left(\frac{1}{2}\right)^{4n} &= \int_0^{\frac{\pi}{4}}3\sin 4n\theta\,d\theta\\ &= \frac{3}{4n}[-\cos 4n\theta]_0^{\frac{\pi}{4}}\\ &= \frac{3}{4n}(-\cos 4n\frac{\pi}{4}+1)\\ &= \frac{3}{4n}(1-(-1)^n)\\ A_n &= \frac{3\cdot 2^{4n}}{4n}(1-(-1)^n)\end{aligned}$$

Therefore the steady-state temperature is

$$u(r,\theta)=\frac{3}{4}\sum_{n=1}^{\infty}\frac{(2r)^{4n}(1-(-1)^n)}{n}\sin 4n\theta \tag{6.45}$$

6.3 Domain of Disk, Semi-Disk, Annulus and Wedge

1. The Dirichlet Problem:

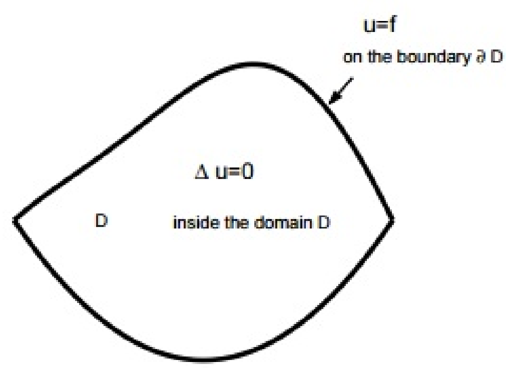


Figure 6.1: The Dirichlet Problem

2. The Neumann Problem:

3. Let a disk of radius, $r=a$. The domains are
   A wedge: $0<r<a; 0<\beta$;
   An annulus: $0<a<r<b$;
   Exterior of a circle: $a<r<b$:

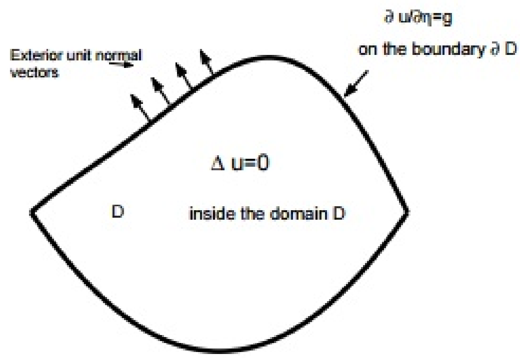


Figure 6.2: The Neumann Problem

6.4 Q5 Jan 2013

a) The solution Laplace equation in polar $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ is given as
$u(r,\theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^{-n} + B_n r^n) \cos n\theta + (C_n r^{-n} + D_n r^n) \sin n\theta)$
State the domain that describe heat distribution in,

i) Disk,

ii) Annulus,

iii) Exterior Domain.

b) Consider a semi circular disk of radius $r = 1$. The temperature on its circumference is governed by the function,

$$f(\theta) = \pi \sin \theta - \sin 2\theta$$

while on its diameter the temperature is zero.

i) Express the problem as a boundary problem in polar variables.

ii) Let the solution be $u(r,\theta) = R(r)T(\theta)$ where $T(\theta)$ the angular component of the solution. By the separation of variables show that the solution, $u(r,\theta)$, within the semi circular disk is $u(r,\theta) = \pi r \sin \theta - r^2 \sin 2\theta$.

Solution

a) Let radius of the disk is $r = b$. The the domain of

Disk:$D = \{0 < r \leq b,\, 0 < \theta \leq 2\pi\}$

Annulus: $D = \{a < r < b,\, 0 < \theta \leq 2\pi\}$ where a= inner circumference of the disk, b=outer circumference of the disk.

Exterior domain: $D = \{b < r < \infty,\ 0\theta \leq 2\pi\}$

b) i) BVP

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,\ 0 < r \leq r,\ 0 < \theta <\leq \pi$$

$s.t$

$$u(r,0) = 0,\ 0 < r \leq 1$$
$$u(r,\pi) = 0,\ 0 < r \leq 1,\ 0 < \theta \leq \pi$$
$$u(1,\theta) = \pi \sin \theta - \sin 2\theta,\ 0 < \theta \leq \pi$$

ii)
Using the separation variable method $u(r,\theta) = R(r)T(\theta)$ and distribute into PDE gives

$$R^{''}T + \frac{1}{r}R^{'}T + \frac{1}{r^2}RT^{''} = 0$$
$$r^2R^{''}T + rR^{'}T + RT^{''} = 0$$
$$\frac{r^2R^{''} + rR^{'}}{R} = -\frac{T^{''}}{T} = \lambda$$

leads to two ODEs,

$$r^2R^{''} + rR^{'} - \lambda R = 0 \tag{6.46}$$
$$T^{''} + \lambda T = 0 \tag{6.47}$$

Translate BC : $u(r,0) = R(r)T(0) = 0 \rightarrow T(0) = 0$, since $R(r) \neq 0$. And BC: $u(r,\pi) = R(r)T(\pi) = 0 \rightarrow T(\pi) = 0$ since $R(r) \neq 0$.
ODE (6.47) together the translated BC above will constitute the Sturm-Liouville problem,

$$T^{''} + \lambda T = 0,\ T(0) = 0,\ T(\pi) = 0 \tag{6.48}$$

The solution of (6.48) are

$$T(\theta) = c_1 + c_2\theta,\ \text{for}\ \lambda = 0 \tag{6.49}$$
$$T(\theta) = c_1 \cosh \alpha\theta = c_2 \sinh \alpha\theta,\ \text{for}\ \lambda = -\alpha^2 < 0 \tag{6.50}$$
$$T(\theta) = c_1 \cos \alpha\theta + c_2 \sin \alpha\theta,\ \text{for}\ \lambda = \alpha^2 > 0 \tag{6.51}$$

Now use BC: $T(0) = 0$ and $T(\pi) = 0$:

$$T(0) = c_1 + c_2(0) \rightarrow c_1 = 0$$
$$T(\theta) = c_1$$
$$T(\pi) = c_1 = 0 \rightarrow c_1 = 0$$

Thus $u(r,\theta) = 0$, a trivial solution.
Eqn (6.50),

$$\begin{aligned} T(0) &= c_1 \cosh(0) = c_2 \sinh(0) \to c_1 = 0 \\ T(\theta) &= c_1 \cosh \alpha\theta \\ T(\pi) &= c_1 \cosh \alpha\phi = 0 \to c_1 = 0 \end{aligned}$$

Thus the solution is trivial.
Next apply BC, $T(0) = 0$ and $T(\pi) = 0$ on (6.51) gives

$$\begin{aligned} T(0) &= c_1 \cos(0) + c_2 \sin \alpha(0) \\ &= c_1(0) + 0 \to c_1 = 0 \\ T(\theta) &= c_1 \sin \alpha\theta \\ T(\pi) &= c_1 \sin \alpha\pi = 0 \end{aligned}$$

For non trivial solution, $c_1 \neq 0$, but $\sin \alpha\pi = 0$. So $\alpha\pi = n\pi \to \alpha = n$. Thus the eigenvalue $\lambda_n = \alpha^2 = n^2,\ n = 1, 2, \ldots$ and the correspondence eigen function of (6.48) ,

$$T(\theta) = c_1 \sin n\theta \tag{6.52}$$

Now we solve the Chauchy-Euler problem (6.46). Let $R = r^m$, $R' = mr^{m-1}$, $R'' = m(m-1)r^{m-2}$ and substitute into (6.46) becomes

$$r^m m(m-1)t^{m-2} + rmr^{m-1} - \lambda r^m = 0$$

$$(m^2 - m + m)r^m = 0 \leftarrow \boxed{\text{for } \lambda_0 = 0}$$

$$m = 0, 0$$

$$\text{So } R(t) = c_1 + c_2 \ln r \leftarrow \boxed{R(r) = c_1 r^m + c_2 r^m \ln r} \tag{6.53}$$

$$r^m m(m-1)r^{m-2} + rmr^{m-1} - n^2 r^m = 0 \leftarrow \boxed{\text{for } \lambda_n = \alpha^2,\ n = 1, 2, \ldots}$$

$$(m^2 - n^2)r^m = 0$$

$$m = n, -n$$

$$\text{So } R(t) = c_1 r^n + c_2 r^{-n} \leftarrow \boxed{\text{real and different roots}} \tag{6.54}$$

For periodic solution $c_2 = 0$ of (7.37) and for bounded solution $c_2 = 0$ of (7.38). A product solution is

$$u_n(r,\theta) = c_1 r^n\, c_1 \sin n\theta \tag{6.55}$$

By superposition principle, the solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta \leftarrow \boxed{A_n = c_1 c_2} \tag{6.56}$$

Now apply BC: $u(1,\theta) = \pi \sin\theta - \sin 2\theta$ becomes

$$u(1,\theta) = \pi \sin\theta - \sin 2\theta = \sum_{0}^{\infty} (\pi \sin\theta - \sin 2\theta) A_n (1^n) \sin n\theta \, d\theta$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} (\pi \sin\theta - \sin 2\theta) \sin n\theta \, d\theta$$

$$\leftarrow \boxed{\text{half range of } \pi \sin\theta - \sin 2\theta \text{ in sine series}}$$

$$= \frac{2}{\pi} \int_0^{\pi} \pi \sin n\theta \sin\theta \, d\theta - \frac{2}{\pi} \int_0^{\pi} \sin 2\theta \sin n\theta \, d\theta$$

For different value of n gives the results is zero (orthogonality).
Now For $n = 1 \to A_1$:

$$\begin{aligned}
A_1 &= \frac{2}{\pi} \int_0^{\pi} \pi \sin^2\theta \, d\theta \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{\pi(1 - \cos 2\theta)}{2} \, d\theta \\
&= [\theta - \frac{1}{2} \sin 2\theta]_0^{\pi} \\
&= (\pi - 0) - 0 = \pi
\end{aligned}$$

For $n = 2 \to A_2$;

$$\begin{aligned}
A_2 &== \frac{2}{\pi} \int_0^{\pi} \sin^2 2\theta \, d\theta \\
&= \frac{2}{\pi} \int_0^{\pi} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta \\
&= \frac{1}{\pi} [\theta - \frac{1}{2} \sin 2\theta]_0^{\pi} \\
&= \frac{1}{\pi} ((1 - 0) - 0) \\
&= 1
\end{aligned}$$

The solution is

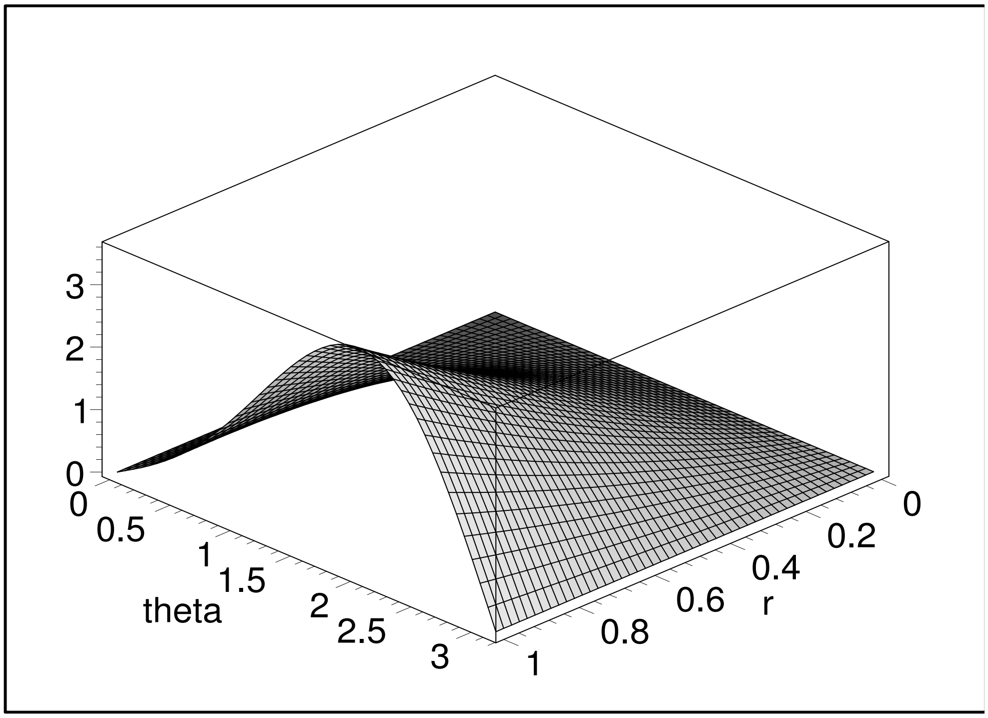
$$\begin{aligned}
u(r,\theta) &= A_1 r \sin n\theta + A_2 r^n \sin n\theta \\
&= \pi \, r \sin\theta - r^2 \sin 2\theta
\end{aligned}$$

6.4.1 Graph of $u(r,\theta) = \pi\, r\, \sin\,\theta - r^2\, \sin\, 2\theta$

```
> restart;
> a:=Pi*r*sin(theta)-r^2*sin(2*theta);
```

$$a := \pi\, r\sin(\theta) - r^2\sin(2\,\theta)$$

```
> plot3d(a,r=0..1,theta=0..Pi,axes=boxed);
```




```
> ?plot3d
```

6.4.2 APR 2011

Q5

a)

$H(\theta)$-problem:Consider $u(r, theta) = R(r)H(\theta)$. Substitute into PDE give

$$r^2R'' + rR' - \lambda R = 0,\ a < r < b \tag{6.57}$$

$$H'' + \lambda H =,\ 0 \le \theta \le \pi \tag{6.58}$$

Eqn (6.58) together with BC constitutes Sturm-Lioville problem

$$H'' + \lambda H = 0,\ H(0) = 0,\ H(\pi) = 0 \tag{6.59}$$

Solve problem (6.59) becomes

$$H(\theta) = c_1 + c_2\theta \leftarrow \boxed{\text{for } \lambda = 0} \tag{6.60}$$

$$H(\theta) = c_1 \cosh \alpha\theta + c_2 \sinh \alpha\theta \leftarrow \boxed{\text{for } \lambda = -\alpha^2 < 0} \tag{6.61}$$

$$H(\theta) = c_1 \cos \alpha\theta + c_2 \sin \alpha\theta \leftarrow \boxed{\text{for } \lambda = \alpha^2 > 0} \tag{6.62}$$

Apply BC:
For (6.60),

$$H(0) = 0 = c_1 + c_2(0) \rightarrow c_1 = 0$$
$$H(\theta) = c_1$$
$$H(\pi) = c_1 = 0 \rightarrow c_1 = 0$$

Thus (6.60) gives trivial solution.
For (6.61),

$$H(0) = c_1(1) + c_2(0) \rightarrow c_1 = 0$$
$$H(\theta) = c_2 \sinh \alpha\theta$$
$$H(\pi) = c_1 \cosh(\alpha\pi) = 0 \rightarrow c_1$$

Similarly, (6.61) gives trivial solution.
Now for (6.62) gives

$$H(0) = c_1(1) + c_2(0) \rightarrow c_1 = 0$$
$$H(\theta) = c_2 \sin \alpha\theta$$
$$H(\pi) = c_2 \sin \alpha\pi = 0$$

For non trivial solution, $c_2 \neq 0$ but $\sin \alpha\pi = 0 \rightarrow \alpha\pi = n\pi \rightarrow \alpha = n$ That is $\lambda = n^2$.So the solution

$$H(\theta) = c_2 \sin n\theta, \ n = 1, 2, \ldots \tag{6.63}$$

<u>$R(r)$-Problem:</u>Now solve Chauchy-Euler (6.57). Let $R = r^m$ gives

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - \lambda r^m = 0$$
$$(m^2 - \lambda)r^m = 0$$
$$m = 0, 0 \leftarrow \boxed{\text{for } \lambda = 0 \text{ and}}$$
$$(m^2 - n)r^m = 0$$
$$m = n, -n \leftarrow \boxed{\text{for } \lambda = n^2}$$

gives the following results

$$R(r) = c_1 + c_2 \ln r \tag{6.64}$$
$$R(r) = c_1 r^n + c_2 r^{-n} \tag{6.65}$$

For bounded to $r = 0$, (6.64) and (6.65) becomes

$$R(r) = c_1 \tag{6.66}$$
$$\tag{6.67}$$

A product solution is

$$u_n(r, \theta) = (c_1 r^n + c_2 r^{-n}) c_2 \sin n\theta \tag{6.68}$$

By superposition principle, the solution is

$$u(r, \theta) = \sum_{n=1}^{\infty} [(c_1 r^n + c_2 r^{-n}) \sin n\theta \tag{6.69}$$

c

Now use BC: $(b, \theta) = 0$: From (6.65),

$$R(b) = c_1 b^n + c_2 b^{-n} = 0 \rightarrow c_2 = -c_1 b^{2n}$$

Substitute into (6.69)

$$u(r, \theta) = \sum_{n=1}^{\infty} \left(c_1 r^n + c_2 r^{-n}\right) c_2 \sin n\theta \tag{6.70}$$
$$= \sum_{n=1}^{\infty} c_1 \left(r^n - \frac{b^{2n}}{r^n}\right) c_2 \sin n\theta \tag{6.71}$$
$$= \sum_{n=1}^{\infty} A_n \left(\frac{r^{2n} - b^{2n}}{r^n}\right) \sin n\theta \leftarrow \boxed{A_n = c_1 c_2} \tag{6.72}$$

d)

$$\begin{aligned}
A_n\left(\frac{a^{2n}-b^{2n}}{a^n}\right) &= \frac{2}{\pi}\int_0^{\pi} 2\theta^2 \sin n\theta \, d\theta \\
&= \frac{4}{\pi}\left[\theta^2 \int \sin n\theta \, d\theta - \int [\int \sin n\theta \, d\theta\right] \frac{d}{\theta}(\theta^2)\, d\theta \\
&= \frac{4}{\pi}\left[\theta^2(-\frac{1}{n}\cos n\theta) + \frac{2}{n}\int \theta \cos n\theta \, d\theta\right] \\
&= -\frac{4}{n\pi}\theta^2 \cos n\theta\Big|_0^{\pi} + \frac{8}{n\pi}\left(\theta \int \cos n\theta \, d\theta - \int [\int \cos n\theta \, d\theta] d\theta\right) \\
&= -\frac{4\pi}{n}((-1)^n) + \frac{8}{n\pi}[\frac{\theta}{n}\sin n\theta\Big|_0^{\pi} + \frac{1}{n^2}\cos n\theta\Big|_0^{\pi}] \\
&= -\frac{4}{n\pi}((-1)^n) + \frac{8}{n^3\pi}((-1)^n - 1) \\
&= \frac{(-8 + 8(-1)^n - 4n^2\pi^2(-1)^n}{n^3\pi} \\
A_n &= \frac{a^n(-8 + 8(-1)^n - 4n^2\pi^2(-1)^n}{n^3\pi(a^{2n} - b^{2n}}
\end{aligned}$$

The solution is given by (6.72) where A_n is the above eqn.

Chapter 7

OCT 2010

7.1 Q5 QCT 2010

Q5 OCT 2010

Consider the Dirichlet problem for steady-state temperature of an annulus,

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0,\ 2<r<4,\ 0<\theta<2\pi$$

a) By using SOV with $u(r,\theta)=R(r)H(\theta)$, show that the solution for $u(r,\theta)$ has the form

$$u(r,\theta)=A_0+B_0\ln r+\sum_{n=1}^{\infty}\left(A_nr^n+B_nr^{-n}\right)\cos n\theta+\left(C_nr^n+D_nr^{-n}\right)\sin n\theta.$$

b) Hence, determine $u(r,\theta)$ if the temperature along the boundaries of the annulus are given by

$$u(2,\theta)=6\cos\theta+10\sin\theta$$
$$u(4,\theta)=15\cos\theta+17\sin\theta$$

Solution

a) By using SOV $u(r,\theta)=R(r)H(\theta)$ and substitute into PDE gives two ODEs

$$rR''+rR'-\lambda R=0 \tag{7.1}$$
$$H''+\lambda H=0 \tag{7.2}$$

We are seeking a solution of the $H(\theta)$-Problem

$$H''+\lambda H=0,\ H(\theta)=H(\theta+2\pi) \tag{7.3}$$

It will form an orthogonal set on the interval $[0, 2\pi]$.
The three possible general solution s of (7.2),

$$H(\theta) = c_1 + c_2\theta, \ \lambda = 0 \tag{7.4}$$
$$H(\theta) = c_1 \cosh \alpha\theta + c_2 \sinh \alpha\theta, \ \lambda = -\alpha^2 < 0 \tag{7.5}$$
$$H(\theta) = c_1 \cos \alpha\theta + c_2 \sin \alpha\theta, \ \lambda = \alpha^2 > 0 \tag{7.6}$$

Solution (7.4) is periodic if $C_2 = 0$. Similarly, solution (7.5) is periodic if $c_1 = c_2 = 0$.The constant solution

$$H(\theta) = c_1 \tag{7.7}$$

can be assigned any period, and so $\lambda = 0$ is an eigenvalue.
Solution (7.6) will be 2π-periodic if $\alpha = n$, where $n = 1, 2, \ldots$.
The eigenvalues of (7.6) are $\lambda_0 = 0$ and $\lambda_n = n^2$, $n = 1, 2, \ldots$.
The correspondence eigenfunctions are

$$H(\theta) = c_1, \ n = 0 \tag{7.8}$$
$$H(\theta) = c_1 \cos n\theta + c_2 \sin n\theta, \ n = 1, 2, \ldots \tag{7.9}$$

$R(r)$-Problem:
The Chauchy-Euler DE (7.1) are

$$R(r) = c_1 + c_2 \ln r, \ n = 0 \tag{7.10}$$
$$R(r) = c_1 r^n + c_2 r^{-n} \ n = 1, 2 \ldots \tag{7.11}$$

Thus product solution $u_n = R(r)H(\theta)$ for Laplace's equation polar are

$$\begin{aligned} u_0 &= (c_1 + c_2 \ln r)(c_1) \\ &= c_1 c_1 + c_1 c_2 \ln r \\ &= A_0 + B_0 \ln r \leftarrow \boxed{A_0 = c_1 c_1 \text{ and } B_0 = c_1 c_2} \\ u_n &= (c_1 r^n + c_2 r^{-n})(c_1 \cos n\theta + c_2 \sin n\theta) \\ &= (c_1 c_1 r^n + c_1 c_2 r^{-n}) \cos n\theta + (c_1 c_2 r^n + c_2 c_2 r^{-n}) \sin n\theta \\ &= (A_n r^n + B_n r^{-n}) \cos n\theta + (C_n r^n + D_n r^{-n}) \sin n\theta \end{aligned}$$

The superposition principle then gives the solution

$$u(r, \theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} \left((A_n r^n + B_n r^{-n}) \cos n\theta + (C_n r^n + D_n r^{-n}) \sin n\theta \right) \tag{7.12}$$

b)
Apply BC: $u(2,\theta) = 6\,cos\,\theta + 10\ \sin\ \theta$

$$u(2,\theta) = 6\ \cos\ \theta + 10\ \sin\ \theta = A_0 + B_0 \ln\ 2 + \sum_{n=1}^{\infty} \big((A_n 2^n + B_n 2^{-n}) \cos\ n\theta + (C_n 2^n + D_n 2^{-n}) \sin\ n$$

$$\begin{aligned} A_0 + B_0\ \ln\ 2 &= \frac{1}{2\pi} \int_0^{2\pi} (6\ \cos\ \theta + 10\ \sin\ \theta) d\theta \\ &= \frac{1}{2\pi} \left[6\ \sin\ \theta - 10\ \cos\ \theta\right]_0^{2\pi} \\ &= \frac{1}{2\pi} (0 - 10(1-1)) = 0 \end{aligned}$$

So

$$A_0 + B_0 \ln\ 2 = 0 \tag{7.13}$$

$$u(2,\theta) = 6\ \cos\ \theta + 10\ \sin\ \theta = \sum_{n=1}^{\infty} (A_n 2^n + B_n 2^{-n}) \cos\ n\theta \tag{7.14}$$

$$A_n 2^n + B_n 2^{-n} = \frac{1}{\pi} \int_0^{2\pi} (6\ \cos\ \theta + 10\ \sin\ \theta) \cos\ n\theta \tag{7.15}$$

$$= \frac{1}{\pi} \left(\int_0^{2\pi} 6\,cos\,\theta\ \cos\ n\theta\, d\theta + \int_0^{2\pi} 10\ \sin\ \theta\ \cos\ n\theta \right) \tag{7.16}$$

$$\text{when } n = 1 \tag{7.17}$$

$$2A_1 + \frac{B1}{2} = \frac{1}{\pi}(6\pi) + 0 \tag{7.18}$$

$$= 6 \tag{7.19}$$

Thus

$$4A_1 + B_1 = 12 \tag{7.20}$$

$$
\begin{aligned}
C_n 2^n + D_n 2^{-n} &= \frac{1}{\pi}\int_0^{2\pi} [6\cos\theta + 10\sin\theta]\sin n\theta\, d\theta \\
2C_1 + \frac{D_1}{2} &= \frac{1}{\pi}\int_0^{2\pi} 6\cos\theta\sin\theta\, d\theta + \frac{1}{\pi}\int_0^{2\pi} 10\sin\theta\sin\theta\, d\theta \\
&= \frac{1}{\pi}\sin^2\theta\Big|_0^{2\pi} + \frac{10}{\pi}\int_0^{2\pi}\sin^2\theta\, d\theta \\
&= 0 + \frac{10}{\pi}\int_0^{2\pi}\frac{1-\cos 2\theta}{2} \\
&= \frac{10}{\pi}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\
&= 10 \\
\text{So } 2C_1 + \frac{D_1}{2} &= 10
\end{aligned}
$$

Thus

$$4C_1 + D_1 = 20 \tag{7.21}$$

Now apply BC: $u(4,\theta) = 15\cos\theta + 17\sin\theta$:

$$
\begin{aligned}
A_0 + B_0 \ln 4 &= \frac{1}{2\pi}\int_0^{2\pi}(15\cos\theta + 17\sin\theta)d\theta \\
&= 0
\end{aligned}
$$

Thus

$$A_0 + B_0 \ln 4 = 0 \tag{7.22}$$

From (5.15) and (7.22) give $A_0 = 0,\ B_0 = 0$.
Next,

$$
\begin{aligned}
A_n 4^n + B_n 4^{-n} &= \frac{1}{\pi}\int_0^{2\pi}(15\cos\theta + 17\sin\theta)\cos n\theta\, d\theta \\
4A_1 + \frac{B_1}{4} &= \frac{1}{\pi}\int_0^{2\pi}(15\cos\theta\cos\theta + 17\sin\theta\cos\theta)d\theta \leftarrow \boxed{\text{Orthogonal series expansion: } n = 1} \\
&= \frac{1}{\pi}(15\pi + 0) \\
&= 15 \\
\text{so, } 4A_1 + \frac{B_1}{4} &= 15
\end{aligned}
$$

Thus

$$16A_1 + B_1 = 60 \tag{7.23}$$

From (7.20) and (7.23); (7.23)-(7.20) gives

$$\begin{aligned}12A_1 &= 48\\ A_1 &= 4\\ 16(4)+B_1 &= 60\\ B_1 &= -4\end{aligned}$$

Finally,

$$\begin{aligned}C_n4^n + D_n4^{-n} &= \frac{1}{\pi}\int_0^{2\pi}(15\cos\theta + 17\sin\theta)\sin n\theta\, d\theta\\ 4C_1 + \frac{D_1}{4} &= \frac{1}{\pi}\int_0^{2\pi}(15\cos\theta\sin\theta + 17\sin\theta\sin\theta)d\theta \leftarrow \boxed{\text{Orthogonal series expansion: } n =}\\ &= \frac{1}{\pi}(0+17\pi)\\ &= 17\\ \text{so, } 4A_1 + \frac{B_1}{4} &= 17\end{aligned}$$

Thus

$$16C_1 + D_1 = 68 \tag{7.24}$$

Apply (7.21) and (7.24), we solve for C_1 and D_1. (7.24)-(7.21) gives

$$\begin{aligned}12C_1 &= 48\\ C_1 &= 4\\ 16(4)+D_1 &= 68\\ D_1 &= 4\end{aligned}$$

The solution is

$$u(r,\theta) = (4r - 4r^{-1})\cos\theta + (4r + 4r^{-1})\sin\theta \tag{7.25}$$

7.1.1 Q4 APR 2010

Q4

Solution

a)
The eigenvalues and eigenfunctions of

$$X'' + \lambda X = 0, X(0) = 0,\ X(1) = 0 \tag{7.26}$$

QUESTION 5

Consider the following nonhomogeneous wave equation

$$u_{tt}(x,t) = 4u_{xx}(x,t) + xe^{-2t}, \quad 0 < x < 1, \quad t > 0.$$
$$u(0,t) = 0$$
$$u(1,t) = 0$$
$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

a) Using the method of eigenfunction expansion, show that $F_n(t) = 2\dfrac{(-1)^{n+1}e^{-2t}}{n\pi}$.

b) Determine the general solution for $u_n(t)$ using the method of undetermined coefficients together with the condition $u(x,0) = 0$.

c) Using the initial condition given, find the specific solution for the boundary value problem.

are found to be

$$\lambda_n = \alpha_n^2 = n^2\pi^2 \text{ and } \sin n\pi x, \; n = 1, 2, 3, \ldots \tag{7.27}$$

If we assume that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin n\pi x$$

$$u_{xx} = \sum_{n=1}^{\infty} u_n(t)(-n^2\pi^2) \sin n\pi x \tag{7.28}$$

$$u_{tt} = \sum_{n=1}^{\infty} u_n''(t) \sin n\pi x \tag{7.29}$$

We can write $F(x,t) = xe^{-2t}$

$$xe^{-2t} = \sum_{n=1}^{\infty} F_n(t)\,\sin\, n\pi x$$

$$F_n(t) = \frac{2}{1}\int_0^1 xe^{-2t}\sin\, n\pi x\, dx \tag{7.30}$$

$$= 2e^{-2t}\int_0^1 x\,\sin\, n\pi x\, dx$$

$$= 2e^{-2t}\left[\frac{-x}{n\pi}\cos\, n\pi x + \frac{1}{n\pi}\int \cos\, n\pi x\, dx\right]$$

$$= 2e^{-2t}\left[-\frac{x}{n\pi}\cos\, n\pi x + \frac{1}{n^2\pi^2}\sin\, n\pi x\right]_0^1 \tag{7.31}$$

$$= 2e^{-2t}\left(\frac{-1}{n\pi}(-1)^n\right)$$

$$= 2e^{-2t}\left(\frac{1}{n\pi}(-1)^{n+1}\right)$$

$$xe^{-2t} = \sum_{n=1}^{\infty}\left(2e^{-2t}\left(\frac{1}{n\pi}(-1)^{n+1}\right)\right)\sin\, n\pi x \tag{7.32}$$

So $F_n(t) = 2\left(\frac{(-1)^{n+1}}{n\pi}\right)e^{-2t}$

b) Substitute (7.28), (7.29) and (7.32) into PDE gives

$$u_{tt} - 4u_{xx} = xe^{-2t} \tag{7.33}$$

$$\sum_{n=1}^{\infty}\left[u_n''(t) + 4n^2\pi^2 u_n(t)\right]\sin\, n\pi x = \sum_{n=1}^{\infty}\left(2e^{-2t}\left(\frac{1}{n\pi}(-1)^{n+1}\right)\right)\sin\, n\pi x \tag{7.34}$$

By equating the coefficients of $\sin\, n\pi x$,

$$u_n''(t) + 4n^2\pi^2 u_n(t) = 2e^{-2t}\left(\frac{1}{n\pi}(-1)^{n+1}\right) \tag{7.35}$$

Solve (7.35) by using undetermined coefficients; Auxiliary equation

$$m^2 + 4n^2\pi^2 = 0$$

$$m = 2n\pi i,\ -2n\pi i \leftarrow \boxed{\text{complex roots}}$$

The complementary solution,

$$u_{nc}(t) = c_1\,\cos\, 2n\pi t + c_2\,\sin\, 2n\pi t \tag{7.36}$$

Particular solution

$$\begin{aligned} u_{np}(t) &= Be^{-2t} \\ u' &= -2Be^{-2t} \\ u'' &= 4Be^{-2t} \end{aligned}$$

Substitute into (7.35) gives

$$4Be^{-2t} + 4n^2\pi^2 Be^{-2t} = \left(\frac{2(-1)^{n+1}}{n\pi}\right)$$

By equating the coefficient of and e^{-2t} becomes

$$B(4 + 4n^2\pi^2) = \left(\frac{2(-1)^{n+1}}{n\pi}\right) \tag{7.37}$$

From (7.37) gives

$$B = \frac{(-1)^{n+1}}{2n\pi(1 + n^2\pi^2)} \tag{7.38}$$

Thus the particular solution

$$u_{np}(t) = \left(\frac{(-1)^{n+1}}{2n\pi(1 + n^2\pi^2)}\right) e^{-2t} \tag{7.39}$$

Hence the general solution

$$\begin{aligned} u_n(t) &= u_{nc} + u_{np} \\ &= c_1 \cos 2n\pi t + c_2 \sin 2n\pi t + \left(\frac{(-1)^{n+1}}{2n\pi(1 + n^2\pi^2)}\right) e^{-2t} \end{aligned}$$

Apply $u(x, 0) = 0$:

$$\begin{aligned} u_n(x, 0) = 0 &= c_1(1) + 0 + \frac{(-1)^{n+1}}{2n\pi(1 + n^2\pi^2)} \\ c_1 &= \frac{(-1)^{n+2}}{2n\pi(1 + n^2\pi^2)} \end{aligned}$$

Hence the general solution

$$u_n(t) = \left(\frac{(-1)^{n+2}}{2n\pi(1 + n^2\pi^2)}\right) \cos 2n\pi t + c_2 \sin 2n\pi t + \left(\frac{(-1)^{n+1}}{2n\pi(1 + n^2\pi^2)}\right) e^{-2t} \tag{7.40}$$

c) The solution,

$$u(x,t) = \sum_{n=1}^{\infty} \left[\left(\frac{(-1)^{n+2}}{2n\pi(1+n^2\pi^2)} \right) \cos 2n\pi t + c_2 \sin 2n\pi t + \left(\frac{(-1)^{n+1}}{2n\pi(1+n^2\pi^2)} \right) e^{-2t} \right] \sin n\pi x \tag{7.41}$$

Apply BC: $u_t(x,0) = 0$:

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[\left(2\frac{(-1)^{n+2}}{2n\pi(1+n^2\pi^2)} \right) (-\sin 2n\pi t) + 2c_2 \cos 2n\pi t - 2\left(\frac{(-1)^{n+1}}{2n\pi(1+n^2\pi^2)} \right) e^{-2t} \right] \sin n\pi$$

$$0 = 2c_2 - 2\left(\frac{(-1)^{n+1}}{2n\pi(1+n^2\pi^2)} \right)$$

$$c_2 = \left(\frac{(-1)^{n+1}}{2n\pi(1+n^2\pi^2)} \right) \tag{7.42}$$

Hence the solution VBP is given by (7.41) where c_2 is given by (7.42).

Chapter 8

Jun 2014

Q5

Given

$$u_{rr} + \frac{1}{r}u - r + \frac{1}{r^2}u - \theta\theta = 0,\ r > c,\ -\pi \leq \theta \leq \pi$$
$$u(c,\theta) = f(\theta),\ -\pi \leq \theta \leq \pi$$

a) Determine the solutions for $R(r)$-problem:
Let $R = r^m$, $R^{'} = mr^{m-1}$, $R^{''} = m(m-1)r^{m-2}$ and sunstitute into

$$r^2R^{''} + rR^{'} - \lambda R = 0 \tag{8.1}$$
$$r^2m(m-1)r^{m-2} + rmr^{m-1} - \lambda r^m = 0 \tag{8.2}$$
$$(m^2 - \lambda)r^m = 0 \tag{8.3}$$
$$\text{Since } r^m \neq 0, \tag{8.4}$$
$$m = 0, 0 \leftarrow \boxed{\text{for } \lambda = 0} \tag{8.5}$$
$$m = n, -n \leftarrow \boxed{\text{for } \lambda = n^2} \tag{8.6}$$

The solution are

$$R(r) = c_1 + c_2 \ln r \tag{8.7}$$
$$R(r) = c_1r^n + c_2r^{-n} \tag{8.8}$$

$R(r)$ are unbounded as r approaches to ∞.
The solution of $R(r)$-problem are

$$R(r) = c_1 \tag{8.9}$$
$$R(r) = c_2r^{-n} \tag{8.10}$$

b)Thus The product solution is

$$u_n(r,\theta) = A_0c_1 + c_2r^{-n}(c_1 \cos n\theta + c_2 \sin n\theta) \tag{8.11}$$

$$= B_0 + r^{-n}(A_n \cos n\theta + B_n \sin n\theta) \leftarrow \boxed{B_0 = A_0c_1,\ A_n = c_1c_2,\ B_n = c_1c_2} \tag{8.12}$$

Hence the general solution is

$$u(r,\theta) = B_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta) \tag{8.13}$$

Now apply BC:$f(\theta) = u(1,\theta) = \pi - |\theta|$

$$\begin{aligned}
B_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - |\theta|)\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{0}(\pi - (-\theta))\,d\theta + \frac{1}{2\pi}\int_{0}^{\pi}(\pi - \theta)\,d\theta \\
&= \frac{1}{2\pi}\left[\pi\theta + \frac{1}{2}\theta^2\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\pi\theta - \frac{1}{2}\theta^2\right]_{0}^{\pi} \\
&= \frac{1}{2\pi}(0 - (\pi^2 + \frac{1}{2}\pi^2) + \frac{1}{2\pi}(\pi^2 - \frac{1}{2}\pi^2) \\
&= \frac{1}{2}\pi
\end{aligned}$$

$$\begin{aligned}
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - |\theta|)\cos n\theta\,d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{0}(\pi + \theta)\cos n\theta\,d\theta + \frac{1}{\pi}\int_{0}^{\pi}(\pi - \theta)\cos n\pi\,d\theta \\
&= -\frac{(-1 + (-1)^2)}{\pi n^2} + -\frac{(-1 + (-1)^2)}{\pi n^2} \\
&= -\frac{2(-1 + (-1)^n)}{\pi n^2}
\end{aligned}$$

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - |\theta|)\sin n\theta\,d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{0}(\pi + \theta)\sin n\theta\,d\theta + \frac{1}{\pi}\int_{0}^{\pi}(\pi - \theta)\sin n\theta\,d\theta \\
&= \frac{(-\pi n + \sin(\pi n))}{\pi n^2} - \frac{(-\pi n + \sin(\pi n))}{\pi n^2} && (8.14) \\
&= 0 && (8.15)
\end{aligned}$$

The solution is give by (8.13) where B_0, A_n and B_n is given by the above .

8.1 Dec 2013

Q5

a) Apply SOV $u(x,y) = X(x)Y(y)$ gives two ODEs

$$X^{''} + \lambda X = 0 \tag{8.16}$$
$$Y^{''} - \lambda Y = 0 \tag{8.17}$$

The Sturm-Liouville associated eqn (8.16) and BC then

$$X^{''} + \lambda X = 0, \; , X(0) = 0, X(\pi) = 0 \tag{8.18}$$

The solutions are

$$X(x) = c_1 + c_2 x \leftarrow \boxed{\lambda = 0} \tag{8.19}$$

$$X(x) = c_3 \cosh \alpha x + c_4 \sinh \alpha x \leftarrow \boxed{\lambda = -\alpha^2 < 0} \tag{8.20}$$

$$X(x) = c_5 \cos \alpha x + c_6 \sin \alpha x \leftarrow \boxed{\lambda = \alpha^2 > 0} \tag{8.21}$$

$$X(0) = 0 = c_1 + C_2(0) \rightarrow c_1$$
$$X(x) = c_2 x$$
$$X(\pi) = 0 = c_2(\pi) \rightarrow c_2 = 0$$

Thus The solution is trivial. Similarly for eqn (8.20). For eqn (8.21)

$$X(0) = 0 = c_5\,(1) + c_6(0) \rightarrow c_5 = 0 \tag{8.22}$$
$$X(x) = c_6 \sin \alpha x \tag{8.23}$$
$$X(\pi) = c_6 \sin \alpha\pi \tag{8.24}$$

For non trivial solution $C_6 \neq 0$ but $\sin \alpha\pi = 0$ that is $\alpha\pi = n\pi \rightarrow \alpha = n$. Thus the solution is

$$X(x) = c_6 \sin nx \tag{8.25}$$

From ODE (8.17) with BC $u(x,1) = 0 \rightarrow Y(1) = 0$